Eur. Phys. J. B **57**, 313–320 (2007)
DOI: 10.1140/epjb/e2007-00162-y

THE EUROPEAN PHYSICAL JOURNAL B

Emergence of bimodality in noisy systems with single-well potential

B. Dybiec[a] and L. Schimansky-Geier[b]

[1] M. Smoluchowski Institute of Physics, and Mark Kac Center for Complex Systems Research, Jagellonian University, ul. Reymonta 4, 30–059 Kraków, Poland
[2] Institut für Physik, Humboldt-Universität zu Berlin, Newtonstrasse 15, 12489 Berlin, Germany

Received 3 March 2007 / Received in final form 27 April 2007
Published online 6 June 2007 –

Abstract. We study the stationary probability density of a Brownian particle in a potential with a single-well subject to the purely additive thermal and dichotomous noise sources. We find situations where bimodality of stationary densities emerges due to presence of dichotomous noise. The solutions are constructed using stochastic dynamics (Langevin equation) or by discretization of the corresponding Fokker-Planck equations. We find that in models with both noises being additive the potential has to grow faster than $|x|$ in order to obtain bimodality. For potentials $\propto|x|$ stationary solutions are always of the double exponential form.

PACS. 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.) – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.50.-r Probability theory, stochastic processes, and statistics – 05.40.Ca Noise

1 Introduction

The stationary probability distribution function (PDF) is one of the most important characteristics of stochastic systems. Its shape results from the joint effect of acting deterministic forces and the dispersion created by the thermal noise. If the forces are given by a potential, the PDF usually possesses a Boltzmann-like structure where locations with vanishing forces correspond to the extrema of the PDF. Additive noise simply broadens the peaks around the maxima of the PDF arising near stable situations of the deterministic systems [1,2].

This correspondence is lost for multiplicative noise where the shape of a stationary PDF can be significantly affected by random changes of the model's parameters [3]. Likewise, colored Gaussian noise with larger correlation times modifies the stationary PDFs and the relation between the deterministic force fields and stationary PDFs becomes case sensitive [4]. Furthermore, the temporal modulations of systems parameters together with thermal noise can significantly improve the system's properties as shown in many models (see [5,6] and references therein). Thus, a Markovian dichotomous modulation of the systems parameters can produce bimodal stationary densities for single-well potentials in the absence of thermal noises [3,7,8].

We intend to study this problem for additive dichotomous noise and thermal noise in more detail. We assume the overdamped one-dimensional Brownian motion in potential force fields with a single stable fixed point and driven by the Markovian dichotomous noise

$$\dot{x}(t) = -V'(x) + \sqrt{2T}\xi(t) + \eta(t). \qquad (1)$$

As it will be outlined, this simple system is able to produce stationary states of a bimodal shape. Thus a particle moving in single-well potentials and subject to dichotomous and thermal noises is a sufficient theoretical setup for emergence of bimodality of stationary states.

For simplicity let us first discuss qualitatively some limits of the stationary density of a linear model with additive Gaussian noise and dichotomous driving. In case without driving the potential is parabolic and the stationary density is Gaussian. Taking adiabatically slow switchings $\eta(t)$ gives a stationary density that is the sum of two Gaussians with the maxima shifted by the values of the driving. Obviously, in this situation we expect bimodality if the amplitudes of the dichotomous driving are larger than the width of the Gaussians which is defined by the value of T. Another limit can be considered as well. Let $T \to 0$. The system is subject only to the dichotomous perturbations. In such a situation, on the one hand, a particle ballistically rolls down to the nearest minimum of the potential, which could be reached in the relaxation time τ_r. On the other hand, the potential is being

[a] e-mail: bartek@th.if.uj.edu.pl
[b] e-mail: alsg@physik.hu-berlin.de

switched between the two configurations with the characteristic switching time τ_s. If the potential can switch between two distinct configurations and $\tau_r \ll \tau_s$, one expects to observe bimodal stationary states, because the particle spends most of the time within the potential minima. The next limit is a fast switching process, $\gamma \to \infty$, for which the correlation time of the dichotomous process $\tau_c \to 0$ (see below for more details) and the dichotomous process can be replaced by the Gaussian white noise [9,10]. In such a situation, the stationary distribution is always Gaussian because the system relaxes to the average minimum of the potential and, in comparison to the case without the dichotomous noise, dispersion around the single maximum is larger.

In the following sections we will use numeric and analytic techniques to derive conditions for a bimodal stationary PDF. Our starting point is a potential with a single-well. We will look for conditions under which emerges the stationary density with a bimodal shape, i.e., with two maxima. The simulations of Langevin equation provide a precise description of motion of a single particle [11]. After averaging over realizations of the process it will be applied in the search for bimodality. The model can be as well associated with a system of two coupled Fokker-Planck equations [3] that describes evolution of the probability density of the process governed by the considered Langevin equation (1). We will use methods of stochastic dynamics and discretization techniques to solve these equations numerically.

2 Model

A dichotomous noise [3,12–14] $\eta(t)$ takes two values. This type of noise is frequently used for modeling of various phenomena in biology [15], physics [16–18] and chemistry [6,19–24]. States of the dichotomous process can be associated with various level of external stimuli, temperature, presence or absence of some external perturbations. Furthermore, the dichotomous process can be used [16, 25] for coupling dynamics of various models between two configurations characterized by distinct values of parameters. Examples of such models include resonant activation [16,25] and ratcheting devices [6,17,18,26].

We assume a system with a variable $x(t)$ which is described by the Langevin equation (1). Therein $\xi(t)$ represents thermal fluctuations which are given by the white Gaussian process with $\langle\xi(t)\xi(s)\rangle = \delta(t-s)$ [27]. The $\eta(t)$ stands for the Markovian dichotomous noise [3] taking one of two possible values $\Delta_\pm$. $\gamma_{\downarrow\uparrow}$ denote transitions rates in a small time interval between the two noise states $\Delta_\pm$. The Markovian dichotomous process $\eta(t)$ remains constant, i.e., $\eta(t) = \Delta_\pm$, for the exponentially distributed time τ, i.e., $P(\tau) \propto \exp(-\gamma_{\downarrow\uparrow}\tau)$. Therefore, the most natural method of generation of the dichotomous process is based on generation of waiting times according to the exponential waiting time distribution [28,29].

In the model, both noises $\xi(t)$ and $\eta(t)$ are assumed to be statistically independent and additive. The Langevin equation (1) describes evolution of a single realization of the stochastic process $\{x(t)\}$ from which further statistics of the process can be calculated. Equation (1) corresponds to the following set of two coupled Fokker-Planck equations [30]

$$\frac{\partial P_\pm(x,t|x_0,t_0)}{\partial t} = \left[\frac{\partial}{\partial x}V'_\pm(x) + T\frac{\partial^2}{\partial x^2}\right] P_\pm(x,t|x_0,t_0) \mp \gamma_\downarrow P_\pm(x,t|x_0,t_0) \pm \gamma_\uparrow P_\mp(x,t|x_0,t_0). \quad (2)$$

$P_\pm(x,t|x_0,t_0)$ is a conditional probability of finding a particle (trajectory) at time t in the vicinity of the point x under the condition that it has started its motion at t_0 from x_0 while the dichotomous noise takes value $\Delta_\pm$. $V'_\pm(x) = V'(x) - \Delta_\pm$, i.e., $V_\pm(x) = V(x) - \Delta_\pm x$, where $V(x)$ is a single-well potential, i.e., in our studies the parabolic potential well $V(x) = x^2/2$. Furthermore, we introduce $P(x,t|x_0,t_0) = P_+(x,t|x_0,t_0) + P_-(x,t|x_0,t_0)$ and in the stationary limit $t - t_0 \to \infty$ the stationary density $P(x) = P_+(x) + P_-(x)$. By altering the dichotomous noise parameters $\Delta_\pm$ and $\gamma_{\uparrow\downarrow}$ it is possible to modify the shape of stationary distributions arising from the model described by equations (1) and (2).

3 Methods of solution and results

The study of the probability densities can be performed by simulation of equation (1) [31] or discretization of equation (2).

Simulations of the Langevin equation are based on the Euler-Mayura integration scheme [31] of equation (1). This method allows inspection of time dependent and stationary probability densities of the studied system. Estimators of stationary probability densities are extracted from the ensemble of $N = 10^6$ trajectories of given length $T_{\max} = 5$ or $T_{\max} = 10$. The length of simulation has been adjusted experimentally to guarantee that for $T_{\max}$ a stationary regime is reached.

The solution of the Fokker-Planck equation [30] can be constructed by discretization of equation (2), which converts the partial differential equation to the discrete Markov chain. By using this discrete Markov chain it is possible to inspect the temporal and stationary behavior of the system described by equation (1).

For the system described by equations (1) and (2) stationary solutions can be easily obtained in the absence of the dichotomous noise ($\Delta_\pm = 0$) [1]

$$P(x) \propto \exp\left[-\frac{V(x)}{T}\right]. \quad (3)$$

As well as in the absence of the thermal noise ($T = 0$) [9,3,32]

$$P(x) \propto \frac{1}{\Delta^2 - [V'(x)]^2}\exp\left[-2\gamma\int^x \frac{V'(x)}{\Delta^2 - [V'(x)]^2}dx\right]. \quad (4)$$

The support of distribution (4) is limited to x such that $\Delta^2 - [V'(x)]^2 > 0$. Furthermore, for the sake of simplicity, it has been assumed that the dichotomous noise is symmetric, i.e., $\gamma_{\downarrow\uparrow} = \gamma$ and $\Delta_{\pm} = \pm\Delta$.

In this symmetric case with $T > 0$ the stationary density $P(x)$ fulfills the following differential equation

$$\begin{aligned}0 &= P'''(x)T^2 + P''(x)2TV'(x)\\ &+P'(x)\left[3TV''(x) + [V'(x)]^2 - \Delta^2 - 2\gamma T\right]\\ &+P(x)\left[2V'(x)V''(x) + TV'''(x) - 2\gamma V'(x)\right]. \quad (5)\end{aligned}$$

The trivial solution of equation (5) is $P(x) \equiv 0$. In general, any non trivial solution of equation (5) depends on three unknown constants, which can be determined by the boundary condition at infinity ($T > 0$)

$$|x| \to \infty \;:\; P(x) \to 0,\; P'(x) \to 0, \quad (6)$$

and normalization of $P_{\pm}(x)$ and consequently of $P(x)$

$$\int_{-\infty}^{\infty} P_{-}(x)dx = \frac{\gamma_{\downarrow}}{\gamma_{\downarrow} + \gamma_{\uparrow}} = 1 - \int_{-\infty}^{\infty} P_{+}(x)dx. \quad (7)$$

In the limit of $T \to 0$ solution of equation (5) is given by equation (4). In the limit $\gamma \to 0$ equations (2) decouple and consequently the system evolves independently in both potentials $V_{\pm}(x)$ leading to $P(x) = C_{-}\exp[-V_{-}(x)/T] + C_{+}\exp[-V_{+}(x)/T]$, where $C_{\pm}$ are determined from the initial conditions, e.g. when $P_{\pm}(x,t|0,0) = \delta(x)/2$ then $C_{-} = C_{+} = C$ and C can be determined due to normalization of the probability density. The limit $\Delta \to 0$ in equation (5) cannot be taken, because it was derived under the assumption that $\Delta \neq 0$. The limit $\Delta \to 0$ can be taken in equation (2) leading to the solution given by equation (3).

In Figure 1 potentials $V_{\pm}(x) = V(x) \mp \Delta x$ are presented which we used for study of stationary densities. We tested numerical methods for the static parabolic potential $V(x) = x^2/2$. Figure 2 presents constructed stationary states. As it is visible in Figure 2 all considered methods produced results that are in full agreement with the theoretical solution $P(x) \propto \exp[-V(x)/T]$. Furthermore, from $P(x)$ the effective non equilibrium potential $V_{\text{eff}} = -\ln[P(x)]$ was estimated, see right panel of Figure 2.

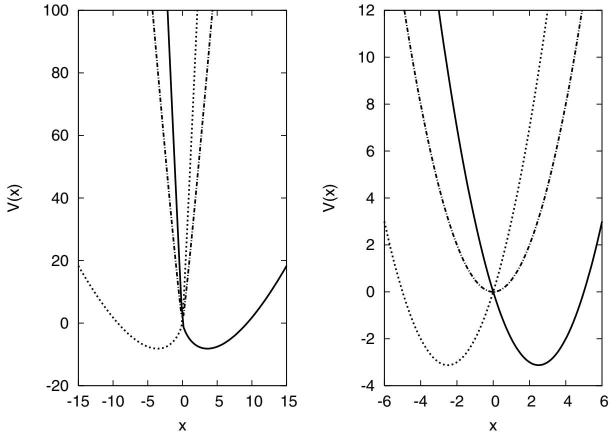

Fig. 1. $V(x)$ (dotted dashed line), $V_{-}(x) = V(x) - \Delta_{-}x$ (dotted line) and $V_{+}(x) = V(x) - \Delta_{+}x$ (solid line) for $V(x) = 20|x|^{1.1}$ with $\Delta_{\pm} = \pm 25$ (left panel) and $V(x) = x^2/2$ with $\Delta_{\pm} = \pm 2.5$ (right panel). The minima of the $|x|^{1.1}$ potential are located at $x \approx \pm 3.59$. The minima of the perturbed parabolic potentials $V_{\pm}(x)$ are located at $\pm\Delta = \pm 2.5$.

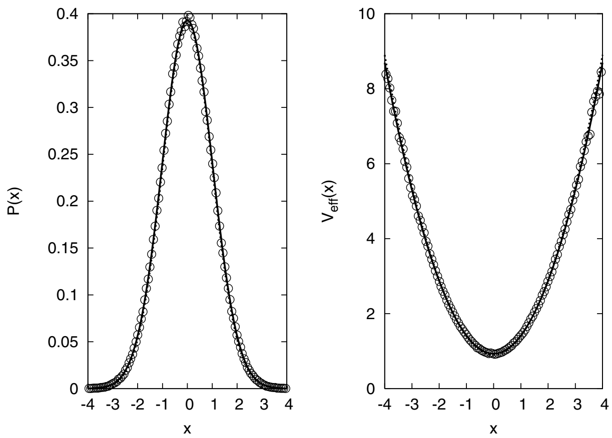

Fig. 2. Stationary states for the static potential $V(x) = x^2/2$ (left panel) and the corresponding effective potential $V_{\text{eff}} = -\ln[P(x)]$ (right panel). Results have been constructed by discretization method (solid line) and Langevin dynamics (∘). Simulations parameters: $\Delta t = 10^{-3}$, $N = 10^6$, $T_{\max} = 10$, $T = 1$ and $N_{\text{bins}} = 200$.

3.1 Special case: V(x) = 0

For $V(x) = 0$ the deterministic force acting on a particle is $-V'_{\pm}(x) = \pm\Delta$ and equation (5) takes the form

$$\begin{aligned}&P'''(x)T^2 - P'(x)(\Delta^2 + 2\gamma T) =\\ &\frac{d}{dx}[P''(x)T^2 - P(x)(\Delta^2 + 2\gamma T)] = 0. \quad (8)\end{aligned}$$

Equation (8) is equivalent to $P''(x)T^2 - P(x)(\Delta^2 + 2\gamma T) = 0$ and it has a solution of the form $P(x) = C_{-}\exp(\lambda_{-}x) + C_{+}\exp(\lambda_{+}x)$ where $\lambda_{\pm} = \pm\sqrt{\Delta^2 + 2\gamma T}/T$. Due to the boundary conditions $P(\pm\infty) = 0$, both constants $C_{\pm} = 0$, and consequently $P(x) \equiv 0$, thus a non trivial stationary PDF does not exist for $V(x) = 0$. The exception is the case of $V(x) = 0$, $T = 0$ with $\gamma \to \infty$ and finite Δ, where an initial condition is persistent. Therefore, for $P_{\pm}(x,t|x_0,0) = \delta(x - x_0)/2$, the stationary solution is $P(x) = \delta(x - x_0)$. For $\gamma \to \infty$, $\Delta^2 \to \infty$ in such a way that Δ^2/γ is constant, the dichotomous noise is equivalent to the Gaussian white noise [9,32]. Thus, the dichotomous driving acts as a thermal noise and consequently probability density is Gaussian with dispersion increasing with time.

The case of $V(x) = 0$ and $T = 0$ leads to the Taylor dispersion model, see [33] and reference therein. In the asymptotic limit, for $t \gg \gamma^{-1}$ and $|x| \ll \Delta \cdot t$ with

$P(x,t|0,0) = \delta(x)$ and $\partial_x P(x,t|0,0) = 0$

$$P(x,t|0,0) \approx \frac{\exp(-x^2\gamma/2\Delta^2 t)}{\sqrt{2\pi\Delta^2 t/\gamma}}. \quad (9)$$

In this context it can be mentioned that non trivial stationary densities, for $V(x) = 0$ and $T > 0$, can be found for a motion of a Brownian particle on a finite interval restricted by two reflecting boundaries that create a permanent box that is shaken by the Markovian dichotomous noise [34]. For the symmetric dichotomous process and symmetric location of reflecting boundaries stationary distributions are also bimodal, with maxima located at the neighborhood of reflecting boundaries. For $\gamma \to \infty$, stationary distributions become uniform [34].

3.2 Special case: $V(x) = |x|$

For $V(x) = |x|$, the dichotomous noise changes slopes of the potential well without moving its minimum, i.e.,

$$V_{\pm}(x) = |x| \mp \Delta x = \begin{cases} x(1\mp\Delta) \text{ for } x \geqslant 0 \\ -x(1\pm\Delta) \text{ for } x < 0 \end{cases}, \quad (10)$$

and consequently minimum of the full potential is located at $x = 0$. Therefore, the dichotomous noise with $V(x) = |x|$ cannot produce bimodal stationary probability densities. Furthermore, to create permanent potential well it is necessary to consider $\Delta < 1$, otherwise the potential switches between two configurations $V_{\pm}(x)$ for which stationary states do not exist.

For $V(x) = |x|$, equation (5) takes the form

$$P'''(x)T^2 \pm P''(x)2T + P'(x)[1-\Delta^2-2\gamma T] \mp P(x)2\gamma = 0. \quad (11)$$

Upper signs correspond to $x > 0$, while lower to $x < 0$.

Due to the symmetry of the dichotomous noise $P(x)$ is an even function of x, i.e., $P(x) = P(-x)$, therefore we introduce

$$P(x) = \sum_{i=1}^{3} C_i \exp(\lambda_i|x|). \quad (12)$$

The unknown constants C_i are determined by the normalization condition $C_1/|\lambda_1| + C_2/|\lambda_2| + C_3/|\lambda_3| = 1/2$ and by the continuity of $P_{\pm}(x)$ at $x = 0$. The characteristic exponents λ_is obey

$$\lambda^3 T^2 + \lambda^2 2T + \lambda[1-\Delta^2-2\gamma T] - 2\gamma = 0. \quad (13)$$

The characteristic determinant of this third order polynomial is $-4T^2\Delta^2 + 8T^2\Delta^4 - 40T^3\Delta^2\gamma - 4\gamma^2T^4 - 4T^2\Delta^6 - 24T^3\Delta^4\gamma - 48T^4\Delta^2\gamma^2 - 32T^5\gamma^3$. The single positive term of the discriminant is $8T^2\Delta^4$, which is easily compensated by other negative terms. Thus, the characteristic determinant is always smaller than zero and, consequently, all roots λ_i are real.

From Vieta's formulas it follows that $\lambda_1\lambda_2\lambda_3 = 2\gamma/T^2 > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = -2/T < 0$. We conclude that equation (13) has always two negative roots, let's say λ_1, λ_2 ($\lambda_1 < \lambda_2$), for every $\gamma > 0$. The third contribution in equation (12) with nonnegative root $\lambda_3 \geqslant 0$ does not obey the boundary conditions and we put $C_3 = 0$.

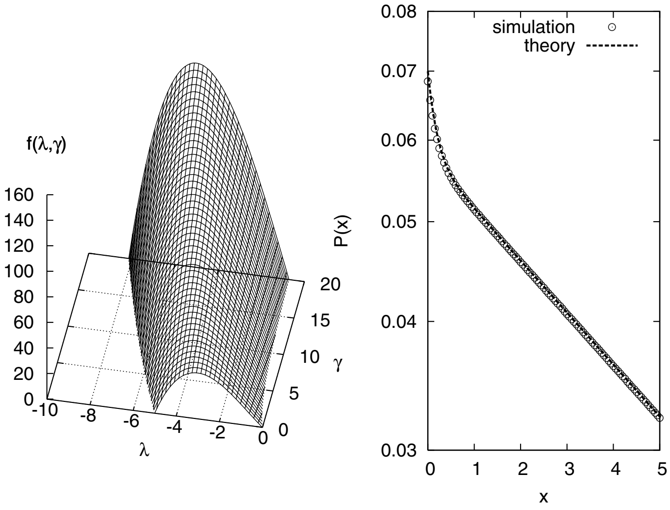

Fig. 3. In the left panel: the surface $f(\lambda,\gamma)$ given by equation (13). Equation (13) has two negative solutions λ_1 and λ_2 ($\lambda_1 < \lambda_2$). For $\gamma \to \infty$ λ_1 tends to $-\infty$, while λ_2 tends to $-1/T$. For $\gamma \to 0$ roots of equation (13) tends to $-1 \pm \Delta$, i.e., they are both negative only for $\Delta < 1$. In the right panel: $P(x)$ for $V(x) = |x|$. Due to the symmetry of $P(x)$, i.e., $P(x) = P(-x)$, results for $x > 0$ are presented only. Please note the logarithmic scale on the ordinate axis. Other model parameters $\Delta = 4$, $\gamma = 1$ and $T = 1$.

For $\gamma = 0$, two negative roots exist if $\Delta < 1$. With the equidistributed initial states $P_{\pm}(x,t|0,0) = \delta(x)/2$, the stationary solution is $P(x) = C_1\exp[-(1-\Delta)|x|/T] + C_2\exp[-(1+\Delta)|x|/T]$, where $C_1 = C_2 = (1-\Delta^2)/4T$. For $\gamma = 0$ and $\Delta \geqslant 1$, a nontrivial stationary solution does not exist.

A more interesting case is $\gamma > 0$. Then the stationary solution exists for arbitrary values of Δ even for $\Delta \geqslant 1$ where states with fixed $\eta(t)$ are unstable. The two characteristic exponents can be found by use of equation (13). Requirement of continuity and normalization leads to

$$C_1 = \frac{|\lambda_1\lambda_2|(T|\lambda_2|-1)}{2[|\lambda_2|(T|\lambda_2|-1) + |\lambda_1|(1-T|\lambda_1|)]}, \quad (14)$$

$$C_2 = \frac{|\lambda_1\lambda_2|(1-T|\lambda_1|)}{2[|\lambda_2|(T|\lambda_2|-1) + |\lambda_1|(1-T|\lambda_1|)]}. \quad (15)$$

In Figure 3 the probability density with $V(x) = |x|$ and parameters $\Delta = 4$, $\gamma = 1$ and $T = 1$ is depicted in the right panel and compared with simulation done by the discretization of equation (2). For $\Delta > 1$, the stable stationary behavior emerges because of the presence of the dichotomous noise. The dichotomous process switches between $\pm\Delta$ forcing the potential to switch between two unstable modes, in every mode the particle would reach arbitrarily large values of x. Switching between both states of the potential induces oppositely directed forces after finite times and, thus, the particle is hindered to reach infinity.

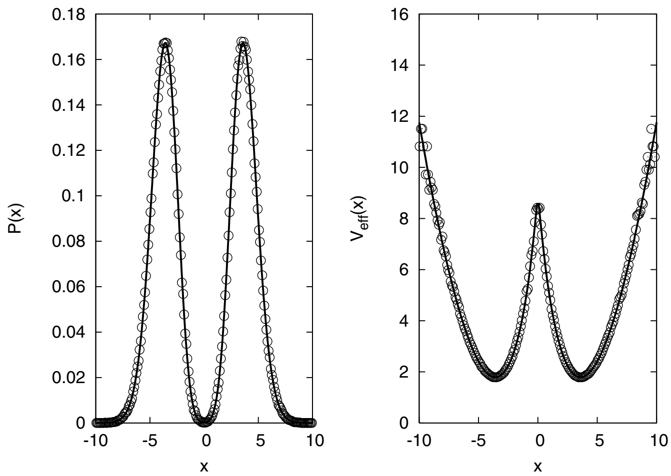

Fig. 4. Stationary solutions for $V(x) = 20|x|^{1.1}$ with $\Delta_\pm = \pm 25$ and corresponding effective non equilibrium potential $V_{\text{eff}} = -\ln[P(x)]$ (right panel) with $\gamma = 0.001$. Results have been constructed by the discretization method (solid line) and by the Langevin dynamics (∘). Simulations parameters: $\Delta t = 10^{-3}$, $N = 10^6$, $T_{\max} = 10$, $T = 1$ and $N_{\text{bins}} = 200$.

The mass of probability is concentrated around $x = 0$ and the situation reminds a Pauli trap.

For $\gamma \to \infty$, $\lambda_1 \to -\infty$ while $\lambda_2 \to -1/T$. For the large absolute value of λ_1, $\lim_{\lambda_1\to-\infty} C_1 = 0$ and $\lim_{\lambda_1\to-\infty} C_2 = |\lambda_2|/2$, see left panel of Figure 3. For $\lambda_1 \to -\infty$, the second characteristic root λ_2 tends to $-1/T$ and, consequently, $C_2 \to 1/(2T)$. Therefore, stationary density tends to its limiting form $P(x) = \exp(-|x|/T)/(2T)$, which is reached for $\gamma \to \infty$. Again, if Δ is becoming infinitely large as $\Delta^2 \propto \gamma$, the dichotomous noise becomes white Gaussian noise and $\lambda_2 \to -1/(T + \Delta^2/2\gamma)$.

In the case $\Delta < 1$ a potential well is present in both states $\pm\,\Delta$. The dichotomous noise only changes the shape of stationary densities. Therefore, the dichotomous driving is not necessary to stabilize stationary states of the inspected model.

3.3 Potential wells of the $|x|^\nu$ type

Equation (5) can be also considered for potentials of the type $V(x) = a|x|^\nu$ $(a > 0)$. In such a situation, the potential well is present for any values of the dichotomous noise $\pm\Delta$ under the condition that $\nu > 1$.

We put $\nu = 1.1$ and hence $V(x) = 20|x|^{1.1}$. The potential is depicted in the left panel of Figure 1. We adjusted the value of the prefactor as $a = 20$ and $\Delta = 25$ in order to accelerate the decay of stationary densities and to decrease the domain with remarkable nonvanishing $P(x)$. Similar shapes of the potential hold for all positive values of a, $\nu > 1$ and Δ. The minima of the considered potential are located at $x \approx \pm 3.59$.

In Figures 4–6 sample stationary PDFs are presented. Due to the dependence of $|x|$ around $x \approx 0$ a cusp is visible. Here, like for the parabolic potential, bimodality

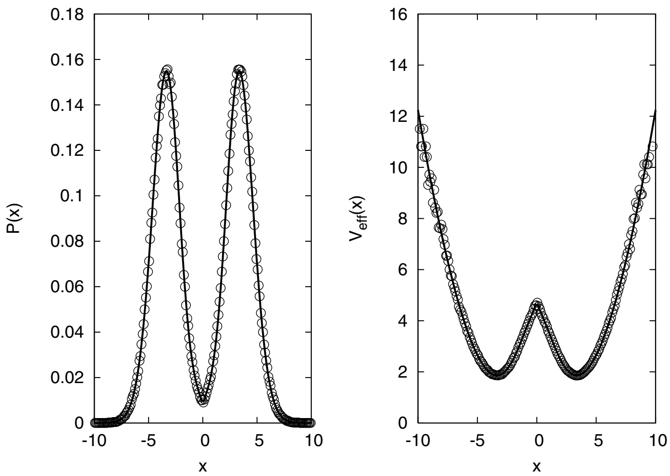

Fig. 5. The same as in Figure 4 for $\gamma = 0.1$.

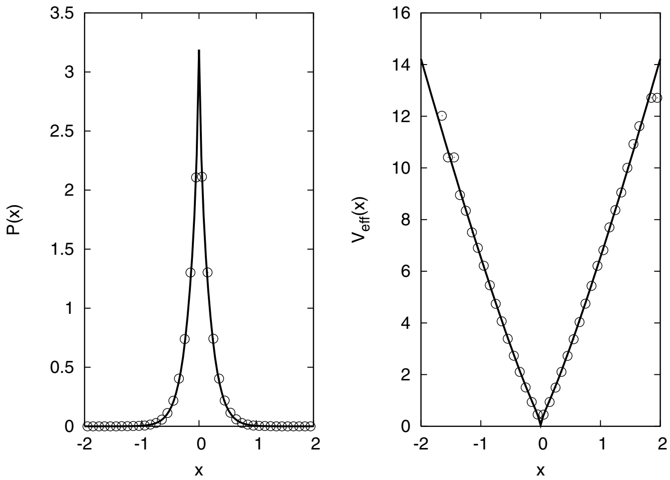

Fig. 6. The same as in Figure 4 for $\gamma = 100$.

of stationary states disappears with increasing γ, see below. The presented solutions were constructed by the discretization technique as well as by the Langevin dynamics.

3.4 Main case: parabolic potential

Figure 7 shows sample trajectories for $V(x) = x^2/2$ with $\Delta_\pm = \pm\Delta = \pm 2.5$. The upper panel corresponds to the situation when a particle is moving in one of the static potentials $V_+(x)$ or $V_-(x)$, which is chosen randomly with equal probabilities. In the middle panel, there is no thermal noise, therefore a particle deterministically rolls down to the minimum of the potential. When the dichotomous process changes its state from $\pm\Delta$ to $\mp\Delta$, the location of the potential minimum changes and consequently the particle starts to roll down to the new position.

In the presence of the thermal noise and the dichotomous process the motion of a particle is irregular, see bottom panel of Figure 7. The level of the irregularity depends on the intensity of the thermal fluctuations. Furthermore, the level of the irregularity can also be affected by the correlation time of the dichotomous noise, i.e., for the small

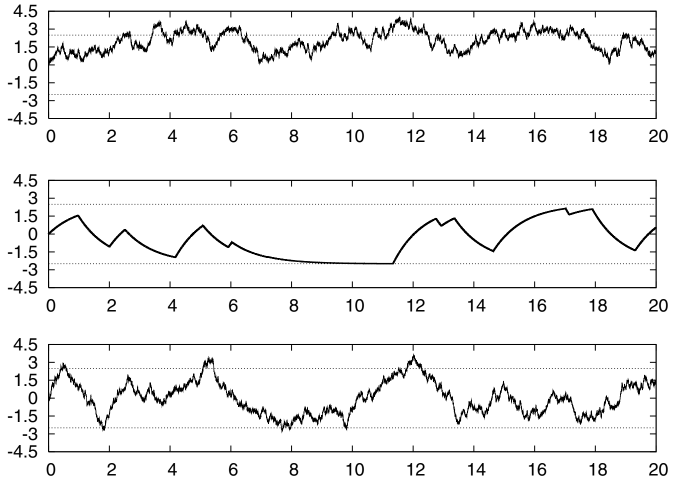

Fig. 7. Sample trajectories for various limiting cases of equation (1) with the parabolic potential: $\gamma_{\downarrow\uparrow} = 0$, $T = 1$ (top panel); $T = 0$ $\gamma_{\downarrow\uparrow} = 0.8$ (middle panel) and $\gamma_{\downarrow\uparrow} = 0.8$, $T = 1$ (bottom panel). For $\Delta_\pm = \pm 2.5$, minima of potentials $V_\pm(x) = x^2/2 - \Delta_\pm x$ are located at ± 2.5.

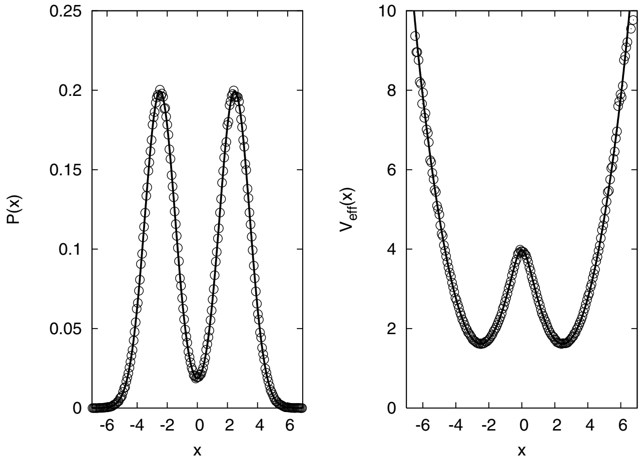

Fig. 8. The stationary solution for $V(x) = x^2/2$ (left panel) and the corresponding effective non equilibrium potential $V_{\text{eff}} = -\ln[P(x)]$ (right panel) with $\Delta_\pm = \pm 2.5$ and $\gamma = 0.001$. Results have been constructed by the discretization method (solid line) and by the Langevin dynamics ($\circ$). Simulations parameters: $\Delta t = 10^{-3}$, $N = 10^6$, $T_{\max} = 5$, $T = 1$ and $N_{\text{bins}} = 200$.

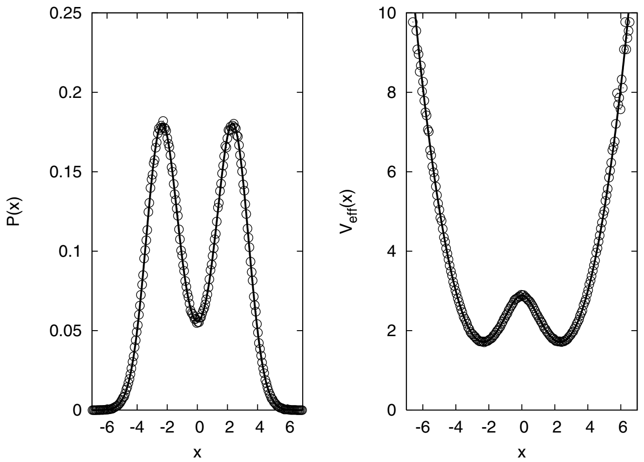

Fig. 9. The same as in Figure 8 for $\gamma = 0.1$.

correlation time τ_c the dichotomous process changes its value more frequently. Transitions between states occur much more rapidly and consequently the motion of the particle is attracted by new positions. Hence, the particle spends a larger fraction of time between minima of the perturbed potential.

Figures 8 and 9 depict the stationary probability density $P(x)$ for the symmetric dichotomous noise characterized by $\Delta = 2.5$ and various switching frequencies γ. For the parabolic potential $V(x) = x^2/2$, the dichotomous noise shifts positions of full potential, $V_\pm(x)$, minima to new locations, see right panel of Figure 1, and consequently bimodality of stationary densities can be recorded. For $\gamma \to 0$, the stationary density is the average of stationary densities for both dichotomous noise configuration, $P(x) \propto [\exp(-V_+(x)) + \exp(-V_+(x))]$, see Figure 8. In the opposite limit, i.e., $\gamma \to \infty$, the stationary density corresponds to the average potential $[V_-(x) + V_+(x)]/2 = V(x) = x^2/2$, see left panel of Figure 2. With the increasing switching frequency γ the minimum separating the two maxima of stationary densities becomes shallower and consequently the bimodal character of stationary densities disappears, see Figures 8 and 9. For $\gamma = 100$, the stationary density reaches its asymptotic limit and is the same as stationary density for the static potential, see left panel of Figure 2.

3.5 Phase diagram

For a given type of the potential $V(x)$ the shape of the stationary states depends on three parameters: T, Δ, γ. On the one hand the bimodality of stationary PDFs is observed for potentials of the type $|x|^{1+\epsilon}$ ($\epsilon > 0$). On the other hand a bimodal shape of stationary solutions can be diminished or destroyed by alternation of the model's parameters T, Δ, γ. Therefore, the natural question is how the phase diagram of the considered model looks like. In general, stationary solutions can be constructed numerically only. Nevertheless, in the simplest cases asymptotic properties of the phase diagram can be inspected analytically, see Figure 10. The condition of multimodality is based on the fact that bimodal stationary solutions for the parabolic potential well subject to the symmetric dichotomous driving have local minima at $x = 0$.

For $\gamma = 0$ (with symmetric initial condition), the stationary solution is the average of stationary states for both configurations of the potential $V_\pm(x) = V(x) \mp \Delta x$. It is bimodal if

$$\left[\Delta^2 + [V'(0)]^2\right]/T - V''(0) > 0. \tag{16}$$

Consequently, for a parabolic potential the bimodality is recorded for $\Delta > \sqrt{T}$.

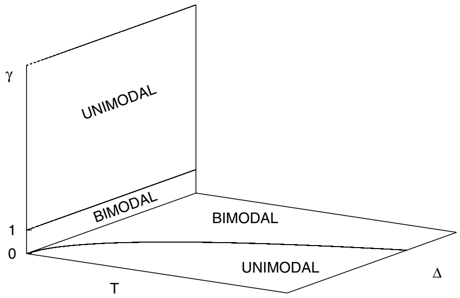

Fig. 10. The schematic sketch of the phase diagram for the parabolic potential $V(x) = x^2/2$. In the limiting cases, the stationary solution is bimodal for $\Delta > \sqrt{T}$ and for $\gamma < 1$. Additional thermal fluctuations broaden the existing stationary states.

For $T = 0$ the stationary solution is given by equation (4) and its support is limited to x such that $\Delta^2 - [V'(x)]^2 > 0$. The solution (4) is bimodal when

$$V'(x)[V''(x) - \gamma] \tag{17}$$

is equal to zero for $x = 0$ and changes its sign from negative $(-)$ to positive $(+)$. Therefore, for the parabolic potential the bimodality is recorded for $\gamma < 1$.

This results confirm our previous expectations, because $V''(0)$ can be associated with the relaxation time within the potential well ($\tau_r \propto 1/\sqrt{V''(0)} = 1$), while γ is related to the switching time of the dichotomous process ($\tau_s \propto 1/\gamma$).

Finally, for $\Delta = 0$, the stationary solution of equation (1) is given by equation (3) and it is always unimodal. The schematic sketch of the phase diagram is depicted in Figure 10.

4 Discussion

Multimodal stationary probability density functions can be observed in various systems. The most natural system is a double-well potential model. For this double-well potential model maximal probability is constructed around the minima of the potential and stationary densities are bimodal. Here we underline that bimodality of stationary densities does not need to be caused by a double-well potential. It can be also produced in single minima potentials. The multimodality could be a consequence of additional dichotomous fluctuations and noise.

A single minima potential with dichotomous and thermal noises provides a sufficient theoretical and experimental setup for the observation of bimodal stationary densities. It is one of the possible frameworks that can explain the origin of bimodality. The bimodality in this approach is observed due to the presence of the dichotomous noise that switches the potential between two distinct configurations. When the minima of the resulting perturbed potentials have different locations, bimodal stationary densities can arise. Therefore, a required potential needs to be steeper than linear. Thus, potentials of the type $|x|^{1+\epsilon}$ ($\epsilon > 0$) are necessary to induce bimodality.

Stationary states can emerge due to the presence of the dichotomous noise in systems which consist of two subsystems for which the stationary states do not exist in each of the two systems taken individually. Namely, if the dichotomous noise switches the potential between two of those configurations, e.g. $|x| \pm 4x$, it can produce a coupled system for which stationary distribution exist. For the linear potential well, i.e., $V(x) = |x|$, the stationary distribution is characterized by two exponents, see right panel of Figure 3, and exists for $\gamma > 0$ and any value of Δ.

Stationary states of stochastic systems driven by dichotomous and thermal noises can be constructed by a direct simulation of the Langevin equation that describes the system's dynamics. Further possibility is to use the associated Fokker-Planck equation, from which the stationary density of the system can be obtained. Here, both approaches were used, leading to an excellent level of agreement between the results obtained in both ways.

In the considered model the emergence of bimodality is the consequence of the combined action of random and deterministic forces. In equation (1) dichotomous noise acts as a two-state stochastic perturbation that changes the shape of the potential experienced by the Brownian particle. Therefore, it is interesting to discuss in some detail the behavior of a similar system in which the dichotomous noise is replaced by the deterministic two-state perturbation, i.e. a symmetric periodic square function. On the one hand, both the periodic square function and the dichotomous noise change the shape of the potential. On the other hand, there are significant differences between both types of perturbation. These differences are especially visible for slow two-state perturbations. For the system perturbed by the dichotomous noise, at a given time t, the perturbation can take any of two allowed values. However, for the system perturbed by the periodic square function, at given time t the potential is in one configuration. This configuration can be easily determined on the basis of the initial condition and parameters of the square function. Therefore, under the assumption that all the initial conditions are deterministic, for the system perturbed by a slowly varying square function the probability density is a periodic function of time [35].

In order to obtain bimodality in the system subject to a periodic two-state perturbation it is necessary to consider a special initial condition. The initial condition should be of the type that introduces a possibility that at a given time t the periodic perturbation can be in any of two allowed states. For example, the appropriate perturbation can be of the form $f(t) = A \times \mathrm{sgn}[\cos(2\pi t/T_\Omega)] \times \mathrm{sgn}(U)$, where U is a random number uniformly distributed over the interval $[-1, 1]$. On the contrary, for fast varying two state perturbations, for both dichotomous and periodic perturbations, resulting distributions are unimodal.

A natural way to extend the considered model with additive dichotomous and thermal noises is to replace the dichotomous process with some process taking more than two values. To record multimodality it is necessary to impose additional conditions on $V(x)$ which are the straightforward extensions of assumptions required for the dichotomous noise. Trichotomous [36] or kangaroo processes [37,38] can be considered as a possible extension of the dichotomous noise.

The research has been supported by the Marie Curie TOK COCOS grant (6th EU Framework Programme under Contract No. MTKD-CT-2004-517186). Computer simulations have been performed using EGEE II (Enabling Grids for E-sciencE) facilities. Additionally, B.D. acknowledges the support from the Foundation for Polish Science and the hospitality of the Humboldt University of Berlin and the Niels Bohr Institute (Copenhagen). Special thanks are directed to E. Gudowska-Nowak, P.F. Góra and I.M. Sokolov for fruitful and inspiring discussions.

References

1. C.W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and Natural Sciences* (Springer Verlag, Berlin, 1983)
2. V.S. Anishchenko, V.V. Astakhov, A.B. Neiman, T.E. Vadisanova, L. Schimansky-Geier, *Nonlinear Dynamics of Chaotic and Stochastic Systems* (Springer Verlag, Berlin, 2003)
3. W. Horsthemke, R. Lefever, *Noise-Inducted Transitions. Theory and Applications in Physics, Chemistry, and Biology* (Springer Verlag, Berlin, 1984)
4. *Noise in Nonlinear Dynamical Systems*, edited by F. Moss, P.V.E. McClintock (Cambridge University Press, Cambridge, 1990)
5. L. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998)
6. P. Reimann, Phys. Rep. **361**, 57 (2002)
7. S. Kabashima, T. Kawakubo, Phys. Lett. A **70**, 375 (1979)
8. J.M. Sancho, M. San Miguel, H. Yamazaki, T. Kawakubo, Physica A **116**, 560 (1982)
9. K. Kitahara, W. Horsthemke, R. Lefever, Phys. Lett. A **70**, 377 (1979)
10. C. Van den Broeck, J. Stat. Phys. **31**, 467 (1983)
11. C. Schmitt, B. Dybiec, P. Hänggi, C. Bechinger, Europhys. Lett. **74**, 937 (2006)
12. I. Bena, C. Van den Broeck, R. Kawai, K. Lindenberg, Phys. Rev. E **68**, 041111 (2003)
13. I. Bena, Int. J. Mod. Phys. B **20**, 2825 (2006)
14. L. Schimansky-Geier, U. Erdmann, N. Komin, Physica A **351**, 51 (2005)
15. R. Mankin, A. Sauga, A. Ainsaar, A. Haljas, K. Paunel, Phys. Rev. E **69**, 061106 (2004)
16. Ch.R. Doering, J.C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992)
17. J. Kula, T. Czernik, J. Łuczka, Phys. Rev. Lett. **80**, 1377 (1998)
18. J. Kula, M. Kostur, J. Łuczka, Chem. Phys. **235**, 27 (1998)
19. A. Fuliński, Phys. Rev. E **52**, 4523 (1995)
20. A.A. Dubkov, N.V. Agudov, B. Spagnolo, Phys. Rev. E **69**, 061103 (2004)
21. B. Spagnolo, A.A. Dubkov, N.V. Agudov, Eur. Phys. J. B **40**, 273 (2004)
22. A.A. Dubkov, B. Spagnolo, Phys. Rev. E **72**, 041104 (2005)
23. B. Gaveau, E. Gudowska-Nowak, R. Kapral, M. Moreau, Phys. Rev. A **46**, 825 (1992)
24. B. Gaveau, E. Gudowska-Nowak, R. Kapral, M. Moreau, Physica A **188**, 443 (1992)
25. B. Dybiec, E. Gudowska-Nowak, Phys. Rev. E **66**, 026123 (2002)
26. P. Reimann, T.C. Elston, Phys. Rev. Lett. **77**, 5328 (1996)
27. N.G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981)
28. B. Dybiec, E. Gudowska-Nowak, P.F. Góra, Int. J. Mod. Phys. C **13**, 1211 (2002)
29. C. Kim, E.K. Lee, P. Talkner, Phys. Rev. E **73**, 026101 (2006)
30. H. Risken, *The Fokker–Planck equation. Methods of Solution and Application* (Springer Verlag, Berlin, 1984)
31. D.J. Higham, SIAM Rev. **43**, 525 (2001)
32. C. Van den Broeck, P. Hänggi, Phys. Rev. A **30**, 2730 (1984)
33. C. Van den Broeck, Physica A **168**, 677 (1990)
34. A.A. Dubkov, P.N. Makhov, B. Spagnolo, Physica A **325**, 26 (2003)
35. P. Jung, Phys. Rep. **234**, 175 (1993)
36. R. Mankin, A. Ainsaar, E. Reiter, Phys. Rev. E **60**, 1374 (1999)
37. H.G. Othmer, S.R. Dunbar, W. Alt, J. Math. Biol. **26**, 263 (1988)
38. M. Kostur, J. Łuczka, Acta Physica Polonica B **30**, 27 (1999)